and an antiperiodic function with a common period is an antiperiodic function with the same period.

Similarly, a sequence $x(n)$ is called an antiperiodic sequence if

$$x(n+N) = -x(n), \tag{33}$$

where N is called the period of the antiperiodic sequence $x(n)$. The product of two antiperiodic sequences with a common period is a periodic sequence. The product of a periodic sequence and an antiperiodic sequence with a common period is an antiperiodic sequence.

When the DWT of a sequence $x(n)$ or the reconstructed sequence is extended, it will form either a periodic or an antiperiodic sequence. In fact, since

$$X_{\mathrm{I}}(m+N) = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} x(n) \sin\left[\frac{\pi}{4} + (m+N)n \cdot \frac{2\pi}{N}\right]$$

$$= \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} x(n) \sin\left[\frac{\pi}{4} + mn \cdot \frac{2\pi}{N}\right],$$

therefore

$$X_{\mathrm{I}}(m+N) = X_{\mathrm{I}}(m). \tag{34}$$

Similarly, it is easy to verify that

$$X_{\mathrm{II}}(m+N) = -X_{\mathrm{II}}(m), \tag{35}$$

$$X_{\mathrm{III}}(m+N) = X_{\mathrm{III}}(m), \tag{36}$$

$$X_{\mathrm{IV}}(m+N) = -X_{\mathrm{IV}}(m). \tag{37}$$

Equations (34) and (36) show that the extended sequences $X_{\mathrm{I}}(m)$ and $X_{\mathrm{III}}(m)$ are periodic sequences with period N. Equations (35) and (37) show that the extended sequences $X_{\mathrm{II}}(m)$ and $X_{\mathrm{IV}}(m)$ are antiperiodic sequences with period N. On the other hand, when the reconstructed sequence $x(n)$ is extended from both sides, it is a periodic sequence with a period N, and will be represented by $x_p(n)$, if it is reconstructed from $X_{\mathrm{I}}(m)$ or $X_{\mathrm{II}}(m)$; or it is

TABLE 1
THE PERIODICITY OR ANTIPERIODICITY OF A DWT[a]

Version j	Original sequence $x(n)$	Transformed sequence $X_j(m)$
I	P	P
II	P	A
III	A	P
IV	A	A

[a] P = periodic sequence, A = antiperiodic sequence.

an antiperiodic sequence with a period N, and will be represented by $x_a(n)$, if it is reconstructed from $X_{\text{III}}(m)$ or $X_{\text{IV}}(m)$. These statements may be verified easily from Equations (21) and (22). Therefore, when DWT-I is performed, it is assumed that both original and transformed data sequences are periodic. When DWT-IV is performed, it is assumed that both sequences are antiperiodic. When DWT-II is performed, it is assumed that the original sequence is periodic, but the transformed sequence is antiperiodic. When DWT-III is performed, it is assumed that the original sequence is antiperiodic, but the transformed sequence is periodic. These conclusions are listed in Table 1.

The conclusions above may be stated in accordance with the symmetry type of the sequence as follows: Whether the length of a sequence is even or odd, and whether the spectrum of that sequence is sampled at integer points or half integer points, the DWT spectrum of an odd symmetry type sequence is periodic, but the DWT spectrum of an even symmetry type sequence is antiperiodic. Since the spatial and frequency domains are equivalent in the W representation [1], the words "sequence" and "spectrum" are interchangeable in the above statement. More explicitly, when DWT-I or DWT-III is performed, the symmetry type chosen for the original sequence is odd, and therefore the transformed sequence is periodic. When DWT-II or DWT-IV is performed, the symmetry type chosen for the original sequence is even, and therefore the transformed sequence is antiperiodic. Considering the mutual inverse relationship of DWT-II and DWT-III we know that the symmetry types for $X_{\text{I}}(m)$ and $X_{\text{II}}(m)$ are odd because they are DWTs of a periodic sequence; and that the symmetric types for $X_{\text{III}}(m)$ and $X_{\text{IV}}(m)$ are even because they are DWTs of an antiperiodic sequence.

When it is necessary, a subscript p or a will be added to $x(n)$, as $x_p(n)$ or $x_a(n)$, to emphasize the periodicity or antiperiodicity of the sequence $x(n)$. $x_p(n)$ and $x_a(n)$ represent the same sequence in the interval from 0 to $N-1$. The difference between them is only in their extension, as discussed above in defining how sequences can be extended.

5. INVARIANCE OF A DWT UNDER A SHIFT OF THE REGION OF SUMMATION

It is well known that the summation of a periodic sequence over one period remains unchanged on a shift of the region of the summation. In other words, if $z(n)$ is a periodic sequence with a period N, then

$$\sum_{n=N_0}^{N_0+N-1} z(n) = \sum_{n=0}^{N-1} z(n).$$

If sequence $x(n)$ and its DWT-j are extended according to Table 1, the elements over which the summation is taken in Equations (5) to (8) and Equations (21) to (24) is either a product of two periodic sequences or a product of two antiperiodic sequences with a common period. Therefore, the summation in Equations (5) through (8) and Equations (21) through (24) is taken on a periodic sequence over one period. It does not depend on which sample is taken as the starting point of the summation. In other words,

$$\sum_{n=N_0}^{N_0+N-1} (\cdot) = \sum_{n=0}^{N-1} (\cdot) \tag{38}$$

holds for all sumations in Equations (5) through (8) and (21) through (24).

6. MIRROR SEQUENCE

$x(-n)$ is called the mirror sequence of $x(n)$, or simply, the mirror of $x(n)$, if $x(n)$ is an odd symmetry type sequence. On the other hand, $x(-n-1)$ is called the mirror sequence of $x(n)$, or simply the mirror of $x(n)$, if $x(n)$ is an even symmetry type sequence. If $x(n)$ is a periodic or an antiperiodic sequence with a period N, the index of the mirror sequence may be removed to the interval from 0 to $N-1$ in accordance with the periodicity or antiperiodicity of the sequence. For example, $x(-n) = x(N-n)$ if $x(n)$ is a periodic sequence, and $x(-n) = -x(N-n)$ if $x(n)$ is an antiperiodic sequence.

The mirror sequences of the four versions of the DWT may be obtained directly from Equations (5) through (8). For example, from Equation (5)

$$X_{\mathrm{I}}(-m) = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} x(n) \sin\left[\frac{\pi}{4} - mn \cdot \frac{2\pi}{N}\right]. \tag{39}$$

Replacing n by $-k$ and shifting the region of addition properly, we obtain

$$X_{\mathrm{I}}(-m)=\sqrt{\frac{2}{N}}\sum_{k=0}^{N-1}x_p(-k)\sin\left[\frac{\pi}{4}+mk\cdot\frac{2\pi}{N}\right], \tag{40}$$

or, using the bi-arrow representation,

$$x_p(-n)\overset{\mathrm{I}}{\Leftrightarrow}X_{\mathrm{I}}(-m). \tag{41}$$

Similarly, we may get

$$x_p(-n-1)\underset{\mathrm{III}}{\overset{\mathrm{II}}{\rightleftharpoons}}X_{\mathrm{II}}(-m), \tag{42}$$

$$x_a(-n)\underset{\mathrm{II}}{\overset{\mathrm{III}}{\rightleftharpoons}}X_{\mathrm{III}}(-m-1), \tag{43}$$

$$x_a(-n-1)\overset{\mathrm{IV}}{\Leftrightarrow}X_{\mathrm{IV}}(-m-1). \tag{44}$$

These results may be stated as follows: The DWT of the mirror of a sequence is the mirror of the DWT of that sequence.

7. THE RELATIONSHIP BETWEEN DFT AND DWT-I

The normalized DFT of a sequence $x(n)$ is defined as

$$X_f(m)=\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}x(n)e^{-imn\cdot 2\pi/N},\qquad m=0,1,\ldots,N-1, \tag{45}$$

where $i=\sqrt{-1}$.

Since

$$e^{-imn\cdot 2\pi/N}=\cos(mn\cdot 2\pi/N)-i\sin(mn\cdot 2\pi/N),$$

$X_f(m)$ may be represented by two summations of cosine and sine terms separately:

$$X_f(m)=\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}x(n)\cos\left(mn\cdot\frac{2\pi}{N}\right)-\frac{i}{\sqrt{N}}\sum_{n=0}^{N-1}x(n)\sin\left(mn\cdot\frac{2\pi}{N}\right),$$

$$m=0,1,\ldots,N-1 \tag{46}$$

But from Equations (5) and (39) we have

$$\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} x(n)\cos\left(mn\cdot\frac{2\pi}{N}\right)$$

$$=\frac{1}{\sqrt{2N}}\sum_{m=0}^{N-1} x(n)\left[\sin\left(\frac{\pi}{4}+mn\cdot\frac{2\pi}{N}\right)+\sin\left(\frac{\pi}{4}-mn\cdot\frac{2\pi}{N}\right)\right]$$

$$=\tfrac{1}{2}\left[X_{\mathrm{I}}(m)+X_{\mathrm{I}}(-m)\right],$$

$$=\tfrac{1}{2}\left[X_{\mathrm{I}}(m)+X_{\mathrm{I}}(N-m)\right], \tag{47}$$

and

$$\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} x(n)\sin\left(mn\cdot\frac{2\pi}{N}\right)$$

$$=\frac{1}{\sqrt{2N}}\sum_{n=0}^{N-1} x(n)\left[\sin\left(\frac{\pi}{4}+mn\cdot\frac{2\pi}{N}\right)-\sin\left(\frac{\pi}{4}-mn\cdot\frac{2\pi}{N}\right)\right]$$

$$=\tfrac{1}{2}\left[X_{\mathrm{I}}(m)-X_{\mathrm{I}}(-m)\right]$$

$$=\tfrac{1}{2}\left[X_{\mathrm{I}}(m)-X_{\mathrm{I}}(N-m)\right]. \tag{48}$$

Therefore,

$$X_f(m)=\tfrac{1}{2}\left[X_{\mathrm{I}}(m)+X_{\mathrm{I}}(N-m)\right]-\frac{i}{2}\left[X_{\mathrm{I}}(m)-X_{\mathrm{I}}(N-m)\right]. \tag{49}$$

Let $[F_N]$ be the normalized DFT matrix

$$[F_N]=\frac{1}{\sqrt{N}}\left[e^{-imn\cdot 2\pi/N}\right], \qquad m,n=0,1,\ldots,N-1. \tag{50}$$

The relationship between DFT and DWT then may be represented in a matrix form

$$[F_N]=[H_N]\left[W_N^{\mathrm{I}}\right], \tag{51}$$

where

$$[H_N] = \left[\begin{array}{c|c|c|c} 1 & & & \\ \hline & \frac{1-i}{2} I_{(N/2)-1} & & \frac{1+i}{2} \bar{I}_{(N/2)-1} \\ \hline & & 1 & \\ \hline & \frac{1+i}{2} \bar{I}_{(N/2)-1} & & \frac{1-i}{2} I_{(N/2)-1} \end{array}\right] \quad \text{if } N \text{ is even,} \tag{52}$$

or

$$[H_N] = \left[\begin{array}{c|c|c} 1 & & \\ \hline & \frac{1-i}{2} I_{(N-1)/2} & \frac{1+i}{2} \bar{I}_{(N-1)/2} \\ \hline & \frac{1+i}{2} \bar{I}_{(N-1)/2} & \frac{1-i}{2} I_{(N-1)/2} \end{array}\right] \quad \text{if } N \text{ is odd.} \tag{53}$$

In Equation (52) and (53), as well as in later matrix representations, a blank submatrix, or blanks in a matrix or in a submatrix, indicates that the elements in that region are zeros; and

$$[\bar{I}] = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{bmatrix} \tag{54}$$

is the opposite identity matrix.

8. RELATIONSHIP AMONG DWT'S

Although four different versions of DWT are defined in Section 3, they are related to each other. Using Equations (5) and (39), $X_{\mathrm{II}}(m)$ of Equation (6) may be expressed as

$$X_{\mathrm{II}}(m) = X_{\mathrm{I}}(m)\cos\frac{m\pi}{N} + X_{\mathrm{I}}(-m)\sin\frac{m\pi}{N}. \tag{55}$$

This relationship may be represented in matrix form as

$$[W_N^{\mathrm{II}}] = [U_N][W_N^{\mathrm{I}}], \tag{56}$$

where

$[U_N] =$

$$
\left[\begin{array}{c|cccc|c|cccc}
1 & & & & & & & & & \\
\hline
 & \cos\frac{\pi}{N} & & & & & & & & \sin\frac{\pi}{N} \\
 & & \cos\frac{2\pi}{N} & & & & & & \sin\frac{2\pi}{N} & \\
 & & & \ddots & & & & \iddots & & \\
 & & & & \cos\frac{(N-2)\pi}{2N} & & \sin\frac{(N-2)\pi}{2N} & & & \\
\hline
 & & & & & 1 & & & & \\
\hline
 & & & & -\sin\frac{(N-2)\pi}{2N} & & \cos\frac{(N-2)\pi}{2N} & & & \\
 & & & \iddots & & & & \ddots & & \\
 & & -\sin\frac{2\pi}{N} & & & & & & \cos\frac{2\pi}{N} & \\
 & -\sin\frac{\pi}{N} & & & & & & & & \cos\frac{\pi}{N}
\end{array}\right]
\tag{57}
$$

If N is even, or

$[U_N] =$

$$
\left[\begin{array}{c|cccccccc}
1 & & & & & & & & \\
\hline
 & \cos\frac{\pi}{N} & & & & & & & \sin\frac{\pi}{N} \\
 & & \cos\frac{2\pi}{N} & & & & & \sin\frac{2\pi}{N} & \\
 & & & \ddots & & & \iddots & & \\
 & & & & \cos\frac{(N-1)\pi}{2N} & \sin\frac{(N-1)\pi}{2N} & & & \\
 & & & & -\sin\frac{(N-1)\pi}{2N} & \cos\frac{(N-1)\pi}{2N} & & & \\
 & & & \iddots & & & \ddots & & \\
 & & -\sin\frac{2\pi}{N} & & & & & \cos\frac{2\pi}{N} & \\
 & -\sin\frac{\pi}{N} & & & & & & & \cos\frac{\pi}{N}
\end{array}\right]
\tag{58}
$$

if N is odd. Likewise, $[W_N^{\mathrm{IV}}]$ is related to $[W_N^{\mathrm{III}}]$ by

$$[W_N^{\mathrm{IV}}] = [V_N][W_N^{\mathrm{III}}], \tag{59}$$

where

$$[V_N] = \begin{bmatrix} \cos\frac{\pi}{2N} & & & & & & & -\sin\frac{\pi}{2N} \\ & \cos\frac{3\pi}{2N} & & & & & -\sin\frac{3\pi}{2N} & \\ & & \ddots & & & \iddots & & \\ & & & \cos\frac{(N-1)\pi}{2N} & -\sin\frac{(N-1)\pi}{2N} & & & \\ & & & \sin\frac{(N-1)\pi}{2N} & \cos\frac{(N-1)\pi}{2N} & & & \\ & & \iddots & & & \ddots & & \\ & \sin\frac{3\pi}{2N} & & & & & \cos\frac{3\pi}{2N} & \\ \sin\frac{\pi}{2N} & & & & & & & \cos\frac{\pi}{2N} \end{bmatrix} \tag{60}$$

if N is even, or

$$[V_N] = \left[\begin{array}{cccc|c|cccc} \cos\frac{\pi}{2N} & & & & & & & & -\sin\frac{\pi}{2N} \\ & \cos\frac{3\pi}{2N} & & & & & & -\sin\frac{3\pi}{2N} & \\ & & \ddots & & & & \iddots & & \\ & & & \cos\frac{(N-2)\pi}{2N} & & -\sin\frac{(N-2)\pi}{2N} & & & \\ \hline & & & & 1 & & & & \\ \hline & & & \sin\frac{(N-2)\pi}{2N} & & \cos\frac{(N-2)\pi}{2N} & & & \\ & & \iddots & & & & \ddots & & \\ & \sin\frac{3\pi}{2N} & & & & & & \cos\frac{3\pi}{2N} & \\ \sin\frac{\pi}{2N} & & & & & & & & \cos\frac{\pi}{2N} \end{array}\right] \tag{61}$$

if N is odd.

Equations (18), (56), and (59) connect each version of the DWT with the other three versions. Since the DWT-I is closely related to the DFT, all versions of the DWT are related to the DFT.

9. DIFFERENT VERSIONS OF DCT AND DST

There are eight versions of the discrete cosine transform and eight versions of the discrete sine transform which relate to those four versions of the DWT introduced in Section 3. The symbols $[C_M^{jE}]$ and $[C_M^{jO}]$ are used to represent matrices of the DCT; $[S_M^{jE}]$ and $[S_M^{jO}]$ are used to represent matrices of the DST, where superscript $j =$ I, II, III or IV represents the version number of DWT to which the DCT or DST relates; superscript E or O represents that the DCT or DST is deduced from the DWT on a data vector which consists of an even or an odd number of data; subscript M represents the order of the matrix. Following this notation, the elements in the mth row and nth column of DCT and DST matrices are defined as follows:

$$[C_{M+1}^{\mathrm{I}E}]_{mn} = \sqrt{\frac{2}{M}}\, k_m k_n \cos\frac{mn\pi}{M}, \qquad m,n = 0,1,\ldots,M, \tag{62}$$

$$[C_M^{\mathrm{II}E}]_{mn} = \sqrt{\frac{2}{M}}\, k_m \cos\frac{m(n+\frac{1}{2})\pi}{M}, \qquad m,n = 0,1,\ldots,M-1, \tag{63}$$

$$[C_M^{\mathrm{III}E}]_{mn} = \sqrt{\frac{2}{M}}\, k_n \cos\frac{(m+\frac{1}{2})n\pi}{M}, \qquad m,n = 0,1,\ldots,M-1, \tag{64}$$

$$[C_M^{\mathrm{IV}E}]_{mn} = \sqrt{\frac{2}{M}} \cos\left[(m+\tfrac{1}{2})(n+\tfrac{1}{2})\frac{\pi}{M}\right], \qquad m,n = 0,1,\ldots,M-1, \tag{65}$$

$$[C_M^{\mathrm{I}O}]_{mn} = \frac{2}{\sqrt{2M-1}}\, k_m k_n \cos\left[mn \cdot \frac{2\pi}{2M-1}\right], \qquad m,n = 0,1,\ldots,M-1, \tag{66}$$

$$[C_M^{\mathrm{II}O}]_{mn} = \frac{2}{\sqrt{2M-1}}\, k_m l_n \cos\left[m(n+\tfrac{1}{2})\frac{2\pi}{2M-1}\right], \qquad m,n = 0,1,\ldots,M-1, \tag{67}$$

$$[C_M^{\mathrm{III}O}]_{mn} = \frac{2}{\sqrt{2M-1}}\, l_m k_n \cos\left[(m+\tfrac{1}{2})n \cdot \frac{2\pi}{2M-1}\right], \qquad m,n = 0,1,\ldots,M-1, \tag{68}$$

$$[C_{M-1}^{\mathrm{IVO}}]_{mn} = \frac{2}{\sqrt{2M-1}} \cos\left[(m+\tfrac{1}{2})(n+\tfrac{1}{2})\frac{2\pi}{2M-1}\right],$$
$$m,n = 0,1,\ldots,M-2, \quad (69)$$

$$[S_{M-1}^{\mathrm{IE}}]_{mn} = \sqrt{\frac{2}{M}} \sin\left(mn\frac{\pi}{M}\right), \qquad m,n = 1,2,\ldots,M-1, \quad (70)$$

$$[S_M^{\mathrm{IIE}}]_{mn} = \sqrt{\frac{2}{M}}\, k_m \sin\left[m\frac{(n-\frac{1}{2})\pi}{M}\right], \qquad m,n = 1,2,\ldots,M, \quad (71)$$

$$[S_M^{\mathrm{IIIE}}]_{mn} = \sqrt{\frac{2}{M}}\, k_n \sin\left[(m-\tfrac{1}{2})\frac{n\pi}{M}\right], \qquad m,n = 1,2,\ldots,M, \quad (72)$$

$$[S_M^{\mathrm{IVE}}]_{mn} = \sqrt{\frac{2}{M}} \sin\left[(m+\tfrac{1}{2})(n+\tfrac{1}{2})\frac{\pi}{M}\right], \quad m,n = 0,1,\ldots,M-1,$$
$$(73)$$

$$[S_{M-1}^{\mathrm{IO}}]_{mn} = \frac{2}{\sqrt{2M-1}} \sin\left[mn\cdot\frac{2\pi}{2M-1}\right], \quad m,n = 1,2,\ldots,M-1, \quad (74)$$

$$[S_{M-1}^{\mathrm{IIO}}]_{mn} = \frac{2}{\sqrt{2M-1}} \sin\left[m(n-\tfrac{1}{2})\frac{2\pi}{2M-1}\right],$$
$$m,n = 1,2,\ldots,M-1, \quad (75)$$

$$[S_{M-1}^{\mathrm{IIIO}}]_{mn} = \frac{2}{\sqrt{2M-1}} \sin\left[(m-\tfrac{1}{2})n\cdot\frac{2\pi}{2M-1}\right],$$
$$m,n = 1,2,\ldots,M-1, \quad (76)$$

$$[S_M^{\mathrm{IVO}}] = \frac{2}{\sqrt{2M-1}}\, l_m l_n \sin\left[(n+\tfrac{1}{2})(n+\tfrac{1}{2})\frac{2\pi}{2M-1}\right],$$
$$m,n = 0,1,\ldots,M-1, \quad (77)$$

where in Equations (62) through (72),

$$k_j = \begin{Bmatrix} \sqrt{2}/2 & \text{if} & j=0 \text{ or } j=M \\ 1 & \text{if} & j\neq 0 \text{ and } j\neq M \end{Bmatrix} \qquad (j = m \text{ or } n), \quad (78)$$

and in Equations (67), (68), and (77),

$$1_j = \begin{Bmatrix} \sqrt{2}/2 & \text{if} & j = M-1 \\ 1 & \text{if} & j \neq M-1 \end{Bmatrix} \qquad (j = m \text{ or } n). \tag{79}$$

The following equations hold for inverse matrices:

$$[C_M^{\mathrm{I}}]^{-1} = [C_M^{\mathrm{I}}], \tag{80}$$

$$[C_M^{\mathrm{II}}]^{-1} = [C_M^{\mathrm{III}}], \tag{81}$$

$$[C_M^{\mathrm{III}}]^{-1} = [C_M^{\mathrm{II}}], \tag{82}$$

$$[C_M^{\mathrm{IV}}]^{-1} = [C_M^{\mathrm{IV}}], \tag{83}$$

$$[S_M^{\mathrm{I}}]^{-1} = [S_M^{\mathrm{I}}], \tag{84}$$

$$[S_M^{\mathrm{II}}]^{-1} = [S_M^{\mathrm{III}}], \tag{85}$$

$$[S_M^{\mathrm{III}}]^{-1} = [S_M^{\mathrm{II}}], \tag{86}$$

$$[S_M^{\mathrm{IV}}]^{-1} = [S_M^{\mathrm{IV}}], \tag{87}$$

where the second superscript, E or O, is omitted because these equations hold for both even and odd cases. For example, Equation (80) is equivalent to both

$$[C_M^{\mathrm{I}E}]^{-1} = [C_M^{\mathrm{I}E}] \quad \text{and} \quad [C_M^{\mathrm{I}O}]^{-1} = [C_M^{\mathrm{I}O}].$$

Some of the transforms defined by Equations (62) through (77) have been defined or discussed by other authors. They are $[S_{M-1}^{\mathrm{I}E}]$ by Jain [8]; $[C_M^{\mathrm{II}E}]$ and $[C_M^{\mathrm{III}E}]$ by Ahmed et al. [9]; $[S_M^{\mathrm{II}E}]$ and $[S_M^{\mathrm{III}E}]$ by Kekre et al. [10]; $[C_M^{\mathrm{I}O}]$ by Pratt [11]; and $[S_{M-1}^{\mathrm{I}O}]$, $[S_{M-1}^{\mathrm{II}O}]$, $[S_{M-1}^{\mathrm{III}O}]$, $[C_{M-1}^{\mathrm{IV}O}]$, $[C_M^{\mathrm{IV}E}]$, and $[S_M^{\mathrm{IV}E}]$ by Jain [12]; and $[C_{M+1}^{\mathrm{I}E}]$ by the authors [13]. But $[C_M^{\mathrm{II}O}]$, $[C_M^{\mathrm{III}O}]$, and $[S_M^{\mathrm{IV}O}]$ have never been reported before.

10. EVEN-ODD TRANSFORM MATRICES

The following orthonormal matrices convert a data vector into its symmetric (even) part and antisymmetric (odd) part. Therefore, they are called even-odd transform (EOT) matrices:

$$[A^{\mathrm{I}}_{2M}] = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c|c|c} \sqrt{2} & & & \\ \hline & I_{M-1} & & \bar{I}_{M-1} \\ \hline & & \sqrt{2} & \\ \hline & \bar{I}_{M-1} & & -I_{M-1} \end{array} \right], \tag{88}$$

$$[A^{\mathrm{II}}_{2M}] = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c} I_M & \bar{I}_M \\ \hline \bar{I}_M & -I_M \end{array} \right], \tag{89}$$

$$[A^{\mathrm{III}}_{2M}] = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c|c|c} \sqrt{2} & & & \\ \hline & I_{M-1} & & -\bar{I}_{M-1} \\ \hline & & \sqrt{2} & \\ \hline & \bar{I}_{M-1} & & I_{M-1} \end{array} \right], \tag{90}$$

$$[A^{\mathrm{IV}}_{2M}] = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c} I_M & -\bar{I}_M \\ \hline \bar{I}_M & I_M \end{array} \right], \tag{91}$$

$$[A^{\mathrm{I}}_{2M-1}] = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c|c} \sqrt{2} & & \\ \hline & I_{M-1} & \bar{I}_{M-1} \\ \hline & \bar{I}_{M-1} & -I_{M-1} \end{array} \right] \tag{92}$$

$$[A^{\mathrm{II}}_{2M-1}] = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c|c} I_{M-1} & & \bar{I}_{M-1} \\ \hline & \sqrt{2} & \\ \hline \bar{I}_{M-1} & & -I_{M-1} \end{array} \right], \tag{93}$$

$$[A^{\mathrm{III}}_{2M-1}] = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c|c} \sqrt{2} & & \\ \hline & I_{M-1} & -\bar{I}_{M-1} \\ \hline & \bar{I}_{M-1} & I_{M-1} \end{array} \right], \tag{94}$$

$$[A^{\mathrm{IV}}_{2M-1}] = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c|c} I_{M-1} & & -\bar{I}_{M-1} \\ \hline & \sqrt{2} & \\ \hline \bar{I}_{M-1} & & I_{M-1} \end{array} \right] \tag{95}$$

where $M > 0$ is an integer; the superscript I, II, III, or IV indicates the version number of DWT with which the EOT is connected; and the subscript indicates the order of the matrix.

$[A^{\mathrm{I}}_{2M}]$ and $[A^{\mathrm{III}}_{2M}]$ convert a data vector with an even number of data points into its odd symmetric and antisymmetric parts. $[A^{\mathrm{II}}_{2M}]$ and $[A^{\mathrm{IV}}_{2M}]$ convert a data vector with an even number of data points into its even symmetric and antisymmetric parts. $[A^{\mathrm{II}}_{2M-1}]$ and $[A^{\mathrm{IV}}_{2M-1}]$ convert a data vector with an odd number of data points into its even symmetric and antisymmetric parts. $[A^{\mathrm{I}}_{2M-1}]$ and $[A^{\mathrm{III}}_{2M-1}]$ convert a data vector with an odd number of data points into its odd symmetric and antisymmetric parts. Therefore, the superscript represents not only the version number, but also the symmetry type to which the data sequence is converted. I or III represent an odd symmetry type, and II or IV represent an even symmetry type.

11. DECOMPOSITION OF THE DWT INTO THE DCT AND DST

All versions of the DWT introduced in Section 3 may be decomposed into the DCT and DST. In the following equations, double bars over a matrix will represent that the indices of both rows and columns are in reversed order:

$$[\bar{\bar{\cdot}}] = [\bar{I}][\cdot][\bar{I}]. \tag{96}$$

The decomposition of different versions of the DWT may be represented by the following equations:

$$[W^{\mathrm{I}}_{2M}] = [A^{\mathrm{I}}_{2M}] \left[\begin{array}{c|c} C^{\mathrm{I}E}_{M+1} & \\ \hline & \bar{\bar{S}}^{\mathrm{I}E}_{M-1} \end{array}\right] [A^{\mathrm{I}}_{2M}], \tag{97}$$

$$[W^{\mathrm{I}}_{2M-1}] = [A^{\mathrm{I}}_{2M-1}] \left[\begin{array}{c|c} C^{\mathrm{I}O}_{M} & \\ \hline & \bar{\bar{S}}^{\mathrm{I}O}_{M-1} \end{array}\right] [A^{\mathrm{I}}_{2M-1}], \tag{98}$$

$$[W^{\mathrm{II}}_{2M}] = [A^{\mathrm{III}}_{2M}]^{-1} \left[\begin{array}{c|c} C^{\mathrm{II}E}_{M} & \\ \hline & \bar{\bar{S}}^{\mathrm{II}E}_{M} \end{array}\right] [A^{\mathrm{II}}_{2M}], \tag{99}$$

$$[W^{\mathrm{II}}_{2M-1}] = [A^{\mathrm{III}}_{2M-1}]^{-1} \left[\begin{array}{c|c} C^{\mathrm{II}O}_{M} & \\ \hline & \bar{\bar{S}}^{\mathrm{II}O}_{M-1} \end{array}\right] [A^{\mathrm{II}}_{2M-1}], \tag{100}$$

$$\left[W_{2M}^{\mathrm{III}}\right]=\left[A_{2M}^{\mathrm{II}}\right]\left[\begin{array}{c|c} C_M^{\mathrm{III}E} & \\ \hline & \bar{\bar{S}}_M^{\mathrm{III}E} \end{array}\right]\left[A_{2M}^{\mathrm{III}}\right], \tag{101}$$

$$\left[W_{2M-1}^{\mathrm{III}}\right]=\left[A_{2M-1}^{\mathrm{II}}\right]\left[\begin{array}{c|c} C_M^{\mathrm{III}O} & \\ \hline & \bar{\bar{S}}_{M-1}^{\mathrm{III}O} \end{array}\right]\left[A_{2M-1}^{\mathrm{III}}\right], \tag{102}$$

$$\left[W_{2M}^{\mathrm{IV}}\right]=\left[A_{2M}^{\mathrm{IV}}\right]^{-1}\left[\begin{array}{c|c} C_M^{\mathrm{IV}E} & \\ \hline & S_M^{\mathrm{IV}E} \end{array}\right]\left[A_{2M}^{\mathrm{IV}}\right], \tag{103}$$

$$\left[W_{2M-1}^{\mathrm{IV}}\right]=\left[A_{2M-1}^{\mathrm{IV}}\right]^{-1}\left[\begin{array}{c|c} C_{M-1}^{\mathrm{IV}O} & \\ \hline & \bar{\bar{S}}_M^{\mathrm{IV}O} \end{array}\right]\left[A_{2M-1}^{\mathrm{IV}}\right]. \tag{104}$$

We shall give the proof of Equation (104) as an example. Let

$$[C]=\left[\cos\left(\left(m+\tfrac{1}{2}\right)\left(n+\tfrac{1}{2}\right)\frac{2\pi}{2M-1}\right)\right], \qquad m,n=0,1,\ldots,M-2, \tag{105}$$

$$[S]=\left[\sin\left(\left(m+\tfrac{1}{2}\right)\left(n+\tfrac{1}{2}\right)\frac{2\pi}{2M-1}\right)\right], \qquad m,n=0,1,\ldots,M-2, \tag{106}$$

$$[I]=[I_{M-1}] \quad \text{and} \quad [\bar{I}]=[\bar{I}_{M-1}]. \tag{107}$$

The order of $[C]$, $[S]$, and $[I]$ is $M-1$, which is omitted for simplicity. $[C]$ and $[S]$ are related to $[C_{M-1}^{\mathrm{IV}O}]$ and $[S_M^{\mathrm{IV}O}]$ by

$$\left[C_{M-1}^{\mathrm{IV}O}\right]=\frac{2}{\sqrt{2M-1}}[C], \tag{108}$$

$$\left[S_M^{\mathrm{IV}O}\right]=\frac{2}{\sqrt{2M-1}}\left[\begin{array}{ccc|c} & & & \sqrt{2}/2 \\ & S & & -\sqrt{2}/2 \\ & & & \vdots \\ \hline \sqrt{2}/2 & -\sqrt{2}/2 & \cdots & (-1)^{M-1}/2 \end{array}\right]. \tag{109}$$

Then we have

$$
\begin{aligned}
&[A^{\mathrm{IV}}_{2M-1}]^{-1}\left[\begin{array}{c|c} C^{\mathrm{IVO}}_{M-1} & \\ \hline & \bar{\bar{S}}^{\mathrm{IVO}}_{M} \end{array}\right][A^{\mathrm{IV}}_{2M-1}] \\
&= \frac{1}{\sqrt{2M-1}}\left[\begin{array}{c|c|c} I & & \bar{I} \\ \hline & \sqrt{2} & \\ \hline -\bar{I} & & I \end{array}\right] \\
&\quad\times\left[\begin{array}{c|c|c} C & & \\ \hline & \frac{(-1)^{M-1}}{2} & \cdots \quad \frac{-\sqrt{2}}{2} \quad \frac{\sqrt{2}}{2} \\ \hline & \begin{matrix} \vdots \\ \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{matrix} & \bar{I}S\bar{I} \end{array}\right]\left[\begin{array}{c|c|c} I & & -\bar{I} \\ \hline & \sqrt{2} & \\ \hline \bar{I} & & I \end{array}\right] \\
&= \frac{1}{\sqrt{2M-1}}\left[\begin{array}{c|c|c} I & & \bar{I} \\ \hline & \sqrt{2} & \\ \hline -\bar{I} & & I \end{array}\right] \\
&\quad\times\left[\begin{array}{c|c|c} C & & -C\bar{I} \\ \hline \frac{\sqrt{2}}{2} \quad \frac{-\sqrt{2}}{2} \quad \cdots & (-1)^{M-1}\frac{\sqrt{2}}{2} & \cdots \quad \frac{-\sqrt{2}}{2} \quad \frac{\sqrt{2}}{2} \\ \hline \bar{I}S & \begin{matrix} \vdots \\ -1 \\ 1 \end{matrix} & \bar{I}S\bar{I} \end{array}\right] \\
&= \frac{1}{\sqrt{2M-1}} \\
&\quad\times\left[\begin{array}{c|c|c} C+S & \begin{matrix} 1 \\ -1 \\ \vdots \end{matrix} & (S-C)\bar{I} \\ \hline 1 \quad -1 \quad \cdots & (-1)^{M-1} & \cdots \quad -1 \quad 1 \\ \hline \bar{I}(S-C) & \begin{matrix} \vdots \\ -1 \\ 1 \end{matrix} & \bar{I}(C+S)\bar{I} \end{array}\right].
\end{aligned}
\tag{110}
$$

But

$$\frac{1}{\sqrt{2M-1}}[C+S] = \frac{1}{\sqrt{2M-1}}\left[\cos\left((m+\tfrac{1}{2})(n+\tfrac{1}{2})\frac{2\pi}{2M-1}\right)\right.$$
$$\left.+\sin\left((m+\tfrac{1}{2})(n+\tfrac{1}{2})\frac{2\pi}{2M-1}\right)\right]$$
$$= \sqrt{\frac{2}{2M-1}}\left[\sin\left(\frac{\pi}{4}+(m+\tfrac{1}{2})(n+\tfrac{1}{2})\frac{2\pi}{2M-1}\right)\right],$$
$$m, n = 0, 1, \ldots, M-2. \quad (111)$$

The right hand side of Equation (111) coincides with the upper left $M-1$ by $M-1$ submatrix of $[W_{2M-1}^{\mathrm{IV}}]$. Likewise,

$$\frac{1}{\sqrt{2M-1}}[S-C][\bar{I}]$$
$$= \frac{1}{\sqrt{2M-1}}\left[\sin\left((m+\tfrac{1}{2})(M-1-n-\tfrac{1}{2})\frac{2\pi}{2M-1}\right)\right.$$
$$\left.-\cos\left((m+\tfrac{1}{2})(M-1-n-\tfrac{1}{2})\frac{2\pi}{2M-1}\right)\right]$$
$$= -\sqrt{\frac{2}{2M-1}}\left[\sin\left(\frac{\pi}{4}-(m+\tfrac{1}{2})(M-1-n-\tfrac{1}{2})\frac{2\pi}{2M-1}\right)\right]$$
$$= \sqrt{\frac{2}{2M-1}}\left[\sin\left(\frac{\pi}{4}+(m+\tfrac{1}{2})(M+n+\tfrac{1}{2})\frac{2\pi}{2M-1}\right)\right],$$
$$m, n = 0, 1, \ldots, M-2,$$

or

$$\frac{1}{\sqrt{2M-1}}[(S-C)\bar{I}] = \sqrt{\frac{2}{2M-1}}\left[\sin\left(\frac{\pi}{4}+(m+\tfrac{1}{2})(n+\tfrac{1}{2})\frac{2\pi}{2M-1}\right)\right],$$
$$m = 0, 1, \ldots, M-2, \quad n = M, M+1, \ldots, 2M-2, \quad (112)$$

which coincides with the upper right $M-1$ by $M-1$ submatrix of $[W_{2M-1}^{\mathrm{IV}}]$.

On the other hand, from Equation (12), the $m+1$th element in the Mth column of $[W_{2M-1}^{\mathrm{IV}}]$ is obtained as

$$
\begin{aligned}
\left[W_{2M-1}^{\mathrm{IV}}\right]_{m,M-1} &= \sqrt{\frac{2}{2M-1}}\sin\left(\frac{\pi}{4}+\left(m+\tfrac{1}{2}\right)\left(M-\tfrac{1}{2}\right)\frac{2\pi}{2M-1}\right)\\
&= \frac{(-1)^m}{\sqrt{2M-1}}, \qquad m=0,1,\ldots,2M-2, \qquad (113)
\end{aligned}
$$

which coincides with the $m+1$th element of the Mth column of the matrix of Equation (110), together with the factor $1/\sqrt{2M-1}$. Therefore, we have proven that the upper $M-1$ rows of the matrix in Equation (110) together with the factor $1/\sqrt{2M-1}$ coincide with the upper $M-1$ rows of $[W_{2M-1}^{\mathrm{IV}}]$.

Repeating the same procedure, we may prove that the lower $M-1$ rows of the matrix in Equation (110) together with the factor $1/\sqrt{2M-1}$ coincide with the lower $M-1$ rows of $[W_{2M-1}^{\mathrm{IV}}]$.

The $n+1$th element of the Mth row of $[W_{2M-1}^{\mathrm{IV}}]$ is

$$
\begin{aligned}
\left[W_{2M-1}^{\mathrm{IV}}\right]_{M-1,n} &= \sqrt{\frac{2}{2M-1}}\sin\left(\frac{\pi}{4}+\left(M-\tfrac{1}{2}\right)\left(n+\tfrac{1}{2}\right)\frac{2\pi}{2M-1}\right)\\
&= \frac{(-1)^n}{\sqrt{2M-1}}, \qquad n=0,1,\ldots,2M-2,
\end{aligned}
$$

which coincides with the $n+1$th element of the Mth row of (110) together with the factor $1/\sqrt{2M-1}$. Therefore, the whole matrix of Equation (110) coincides with $[W_{2M-1}^{\mathrm{IV}}]$, and Eq. (104) is proven.

The proof of all equations from (97) to (103) is similar to the proof given above and therefore is omitted.

12. ANTIPERIODIC CONVOLUTION AND ANTIPERIODIC CORRELATION

Representing the periodic convolution of two periodic sequences $x_p(n)$ and $y_p(n)$ with a common period N by $x_p(n) * y_p(n)$, we define it as

$$
x_p(n) * y_p(n) = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} x_p(k)y_p(n-k). \qquad (114)
$$

Representing the cross correlation of $x_p(n)$ and $y_p(n)$ by $x_p(n) \circledast y_p(n)$, it is defined as

$$x_p(n) \circledast y_p(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_p(k) y_p(n+k). \tag{115}$$

The definitions (114) and (115) differ from the conventional definitions of the periodic convolution and the cross correlation of two periodic sequences by a factor of $1/\sqrt{N}$.

It is obvious from (114) and (115) that both $x_p(n) * y_p(n)$ and $x_p(n) \circledast y_p(n)$ are periodic sequences in n. Their period is N too. On the other hand, since the product sequences $x_p(k)y_p(n-k)$ and $x_p(k)y_p(n+k)$ are periodic sequences and the summations in (114) and (115) are taken over a period N, according to Section 5, the region of summation may be shifted. Substituting $j = n - k$ in (114), we have

$$\begin{aligned} x_p(n) * y_p(n) &= \frac{1}{\sqrt{N}} \sum_{j=1}^{N-1+n} x_p(n-j) y_p(j) \\ &= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} y_p(j) x_p(n-j) \\ &= y_p(n) * x_p(n), \end{aligned} \tag{116}$$

which shows that the periodic convolution obeys the commutative law.

Substituting $j = n + k$ in Equation (115), we have

$$\begin{aligned} x_p(n) \circledast y_p(n) &= \frac{1}{\sqrt{N}} \sum_{j=n}^{N-1+n} x_p(j-n) y_p(j) \\ &= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} y(j) x_p[-(n-j)] \\ &= y_p(n) * x_p(-n) \\ &= x_p(-n) * y_p(n). \end{aligned} \tag{117}$$

Therefore, the cross correlation of two periodic sequences $x_p(n)$ and $y_p(n)$

may be treated as the convolution of the mirror sequence of $x_p(n)$ and the sequence $y_p(n)$.

If $x(n)$ and $y(n)$ are two antiperiodic sequences represented by $x_a(n)$ and $y_a(n)$, we may define the antiperiodic convolution of them by

$$x_a(n) * y_a(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_a(k) y_a(n-k), \tag{118}$$

and the antiperiodic cross correlation of them by

$$x_a(n) \circledast y_a(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_a(k) y_a(n+k). \tag{119}$$

It is easy to see from the definitions (118) and (119) that both $x_a(n) * y_a(n)$ and $x_a(n) \circledast y_a(n)$ are antiperiodic sequences in n. However, since both $x_a(k)$ and $y_a(n-k)$ or $y_a(n+k)$ are antiperiodic sequences in k, the product sequences $x_a(k)y_a(n-k)$ and $x_a(k)y_a(n+k)$ are periodic sequences with respect to k. As in the derivation of Equations (127) and (128), we may also obtain

$$x_a(n) * y_a(n) = y_a(n) * x_a(n), \tag{120}$$

and

$$x_a(n) \circledast y_a(n) = x_a(-n) * y_a(n). \tag{121}$$

Equation (120) shows that the antiperiodic convolution also obeys the commutative law. Equation (121) shows that the antiperiodic cross correlation may be treated as an antiperiodic convolution.

Although $x_a(n)$ and $y_a(n)$ may represent the same sequences $x_p(n)$ and $y_p(n)$ in the interval 0 to $N-1$, respectively, $x_a(n) * y_a(n)$ and $x_a(n) \circledast y_a(n)$ are different sequences from $x_p(n) * y_p(n)$ and $x_p(n) \circledast y_p(n)$ even inside the interval 0 to $N-1$. But, on the other hand, they possess some common properties, as we shall show.

We have shown that correlation may be treated as convolution. Therefore, we shall only discuss the W representation of the convolution theorem.

13. CONVOLUTION THEOREMS

Before the discussion of convolution theorems, we define a special kind of product of two sequences. The following sequence $Z(m)$ is called the mirror product of the sequences $X(m)$ and $Y(m)$:

$$Z(m) = \tfrac{1}{2}\{X(m)[Y(m) + Y(-m)] + X(-m)[Y(m) - Y(-m)]\} \tag{122}$$

if both $X(m)$ and $Y(m)$ are of odd symmetry type, or

$$Z(m) = \tfrac{1}{2}\{X(m)[Y(m) + Y(-m-1)] + X(-m-1)[Y(m) - Y(-m-1)]\} \tag{123}$$

if both $X(m)$ and $Y(m)$ are of even symmetry type. The mirror product of $X(m)$ and $Y(m)$ will be denoted by $X(m) \odot Y(m)$. It obeys the commutative, distributive, and associative laws and possesses some special properties that are shown in the Appendix.

The convolution theorem may be stated briefly as follows: The DWT of the convolution of two sequences equals the mirror product of the DWTs of those two sequences. More explicitly, the convolution theorem may be represented by the following expressions:

$$x_p(n) * y_p(n) \overset{\mathrm{I}}{\Leftrightarrow} X_{\mathrm{I}}(m) \odot Y_{\mathrm{I}}(m), \tag{124}$$

$$x_p(n-1) * y_p(n-1) \overset{\mathrm{I}}{\Leftrightarrow} X_{\mathrm{II}}(m) \odot Y_{\mathrm{II}}(m), \tag{125}$$

$$x_p(n) * y_p(n) \underset{\mathrm{III}}{\overset{\mathrm{II}}{\rightleftharpoons}} X_{\mathrm{I}}(m) \odot Y_{\mathrm{II}}(m) = X_{\mathrm{II}}(m) \odot Y_{\mathrm{I}}(m), \tag{126}$$

$$x_a(n) * y_a(n) \underset{\mathrm{II}}{\overset{\mathrm{III}}{\rightleftharpoons}} X_{\mathrm{III}}(m) \odot Y_{\mathrm{III}}(m), \tag{127}$$

$$x_a(n-1) * y_a(n-1) \underset{\mathrm{II}}{\overset{\mathrm{III}}{\rightleftharpoons}} X_{\mathrm{IV}}(m) \odot Y_{\mathrm{IV}}(m), \tag{128}$$

$$x_a(n) * y_a(n) \overset{\mathrm{IV}}{\Leftrightarrow} X_{\mathrm{III}}(m) \odot Y_{\mathrm{IV}}(m) = X_{\mathrm{IV}}(m) \odot Y_{\mathrm{III}}(m), \tag{129}$$

where in (125) and (128), dropping the subscript p or a, $x(n-1) * y(n-1)$ is defined as

$$x(n-1) * y(n-1) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x(k) y(n-1-k). \tag{130}$$

The procedure of proof of each of Equations (124) through (129) is the same one. Therefore, we shall only show the proof of Equation (127) as an example.

Let

$$z_a(n) = x_a(n) * y_a(n)$$
$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_a(k) y_a(n-k). \tag{131}$$

The DWT-III of $z_a(n)$ is

$$Z_{\mathrm{III}}(m) = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} z_a(n) \sin\left[\frac{\pi}{4} + (m+\tfrac{1}{2})n \cdot \frac{2\pi}{N}\right]$$
$$= \frac{\sqrt{2}}{N} \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} x_a(k) y_a(n-k) \sin\left[\frac{\pi}{4} + (m+\tfrac{1}{2})n \cdot \frac{2\pi}{N}\right]. \tag{132}$$

Let $j = n - k$. Substituting $n = j + k$ into Equation (132), we obtain

$$Z_{\mathrm{III}}(m) = \frac{\sqrt{2}}{N} \sum_{j=-k}^{N-1-k} \sum_{k=0}^{N-1} x_a(k) y_a(j) \sin\left[\frac{\pi}{4} + (m+\tfrac{1}{2})(j+k)\frac{2\pi}{N}\right]$$
$$= \sqrt{\frac{2}{N}} \sum_{j=-k}^{N-1-k} y_a(j) \sin\left[\frac{\pi}{4} + (m+\tfrac{1}{2})j\frac{2\pi}{N}\right]$$
$$\times \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_a(k) \cos\left[(m+\tfrac{1}{2})k\frac{2\pi}{N}\right]$$
$$+ \sqrt{\frac{2}{N}} \sum_{j=-k}^{N-1-k} y_a(j) \cos\left[\frac{\pi}{4} + (m+\tfrac{1}{2})j \cdot \frac{2\pi}{N}\right]$$
$$\times \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_a(k) \sin\left[(m+\tfrac{1}{2})k - \frac{2\pi}{N}\right]. \tag{133}$$

Since

$$\cos\left[\frac{\pi}{4} + (m+\tfrac{1}{2})j \cdot \frac{2\pi}{N}\right] = \sin\left[\frac{\pi}{4} - (m+\tfrac{1}{2})j \cdot \frac{2\pi}{N}\right], \tag{134}$$

shifting the region of summation of j to the region from 0 to $N-1$ according to Section 5, and using Equations (134), (7), and (45), we obtain

$$Z_{\mathrm{III}}(m)=Y_{\mathrm{III}}(m)\cdot\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}x(k)\cos\left[\left(m+\tfrac{1}{2}\right)k\cdot\frac{2\pi}{N}\right]$$

$$+Y_{\mathrm{III}}(-m-1)\cdot\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}x(k)\sin\left[\left(m+\tfrac{1}{2}\right)k\cdot\frac{2\pi}{N}\right]. \quad (135)$$

On the other hand,

$$X_{\mathrm{III}}(m)+X_{\mathrm{III}}(-m-1)=\sqrt{\frac{2}{N}}\sum_{k=0}^{N-1}x(k)\left\{\sin\left[\frac{\pi}{4}+\left(m+\tfrac{1}{2}\right)k\cdot\frac{2\pi}{N}\right]\right.$$

$$\left.+\sin\left[\frac{\pi}{4}-\left(m+\tfrac{1}{2}\right)k\cdot\frac{2\pi}{N}\right]\right\}$$

$$=\frac{2}{\sqrt{N}}\sum_{k=0}^{N-1}x(k)\cos\left[\left(m+\tfrac{1}{2}\right)k\cdot\frac{2\pi}{N}\right] \quad (136)$$

and

$$X_{\mathrm{III}}(m)-X_{\mathrm{III}}(-m-1)=\sqrt{\frac{2}{N}}\sum_{k=0}^{N-1}x_a(k)\left\{\sin\left[\frac{\pi}{4}+\left(m+\tfrac{1}{2}\right)k\cdot\frac{2\pi}{N}\right]\right.$$

$$\left.-\sin\left[\frac{\pi}{4}-\left(m+\tfrac{1}{2}\right)k\cdot\frac{2\pi}{N}\right]\right\}$$

$$=\frac{2}{\sqrt{N}}\sum_{k=0}^{N-1}x(k)\sin\left[\left(m+\tfrac{1}{2}\right)k\cdot\frac{2\pi}{N}\right]. \quad (137)$$

Substituting (136) and (137) into (135) and noticing the definition (123) of the mirror product of an even symmetry type sequence, we obtain

$$Z_{\mathrm{III}}(m)=Y_{\mathrm{III}}(m)\odot X_{\mathrm{III}}(m). \quad (138)$$

Interchanging $X_{\mathrm{III}}(m)$ and $Y_{\mathrm{III}}(m)$ according to Equation (A-2) (in the Appendix) and using the bi-arrow notation yields Equation (127).

14. CONCLUDING COMMENTS

Four versions of the DWT have been introduced. The decomposition of the DWT leads to the DCT and DST. Each version of the DWT relates to two versions of both DCT and DST by two types of EOT. Therefore, eight versions of both DCT and DST are related to four different versions of the DWT. The antiperiodic function, antiperiodic sequence and antiperiodic convolution have been defined. They provide more flexibility for digital harmonic analysis. The convolution theorem holds for both the periodic and antiperiodic cases.

The decomposition of the DWT and DFT into the DCT and DST indicates a new direction for finding fast algorithms for the DWT as well as for the DFT by continuously decomposing the DCT and DST matrices. Since the DCT and DST matrices are real, only real arithmetic will be involved in the new algorithms. This will be the topic of a future paper.

APPENDIX. SUMMARY OF MIRROR PRODUCT ALGEBRA

A function $h(x)$ is called the mirror product of functions $f(x)$ and $g(x)$ if

$$h(x)=\tfrac{1}{2}\{f(x)[g(x)+g(-x)]+f(-x)[f(x)-f(-x)]\}. \quad \text{(A-1)}$$

The mirror product of $f(x)$ and $g(x)$ will be denoted by $f(x)\odot g(x)$. It possesses the following properties:

$$f(x)\odot g(x)=g(x)\odot f(x) \qquad \text{(commutative law)}, \qquad \text{(A-2)}$$

$$f(x)\odot[g(x)+h(x)]=f(x)\odot g(x)+f(x)\odot h(x) \qquad \text{(distributive law)}, \qquad \text{(A-3)}$$

$$[f(x)\odot g(x)]\odot h(x)=f(x)\odot[g(x)\odot h(x)] \qquad \text{(associative law)}, \quad \text{(A-4)}$$

$$c\odot f(x)=cf(x), \qquad \text{(A-5)}$$

$$f_e(x)\odot g(x)=f_e(x)g(x), \qquad \text{(A-6)}$$

$$f_o(x)\odot g(x)=f_o(x)g(-x), \qquad \text{(A-7)}$$

$$f(x)\odot f(-x)=\tfrac{1}{2}\left[f(x)^2+f(-x)^2\right], \qquad \text{(A-8)}$$

where c is a constant, $f_e(x)$ is an even function, $f_o(x)$ is an odd function, and

$f(x) \odot f(-x)$ may be called the amplitude function of $f(x)$. The proof of these properties is straightforward and therefore is omitted.

REFERENCES

1 Z. Wang, Harmonic analysis with a real frequency function. I. Aperiodic case, *Appl. Math. Comput.* 9:53–73 (1981).

2 Z. Wang, Harmonic analysis with a real frequency function. II. Periodic and bounded cases, *Appl. Math. Comput.* 9:153–163 (1981).

3 Z. Wang, Harmonic analysis with a real frequency function. III. Data sequence, *Appl. Math. Comput.* 9:245–255 (1981).

4 H. C. Andrews and W. K. Pratt, Fourier transform coding of image, in *Hawaii International Conference on System Science*, Jan. 1968, pp. 677–679.

5 H. C. Andrews and W. K. Pratt, Television bandwidth reduction by encoding spatial frequencies, *Soc. Motion Picture and Television Engrs.* 77:1279–1281 (1968).

6 H. C. Andrews and W. K. Pratt, Transform image coding, in *Proceedings of Computer Processing in Communications*, Polytechnic Press, New York, 1969, pp. 63–84.

7 W. K. Pratt, J. Kane, and H. C. Andrews, Hadamard transform image coding, *Proc. IEEE* 57:58–68 (1969).

8 A. K. Jain, A fast Karhunen-Loeve transform for a class of stochastic processes, *IEEE Trans. Comm.* COM-24:1023–1029 (1976).

9 H. Ahmed, T. Natarajan, and K. R. Rao, Discrete cosine transform, *IEEE Trans. Comput.* C-23:90–93 (1974).

10. H. B. Kekre and J. K. Solanki, Comparative performance of various trigonometric unitary transforms for transform image coding, *Internat J. Electron.* 44:305–315 (1978).

11 W. K. Pratt, *Digital Image Processing*, Wiley, New York, 1978, Chapters 10, 23.

12 A. K. Jain, A sinusoidal family of unitary transform, *IEEE Trans.* PAMI-1:356–365 (1979).

13 Z. Wang and B. R. Hunt, The discrete cosine transform—a new version, in *Proceedings of 1983 International Conference on Acoustics, Speech, and Signal Processing*.

Modeling Behavior in Competition: The Analytic Hierarchy Process

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ABSTRACT

This paper offers an approach for dealing with prediction of the outcome of World Chess Championship matches based on players experience and attitude towards the game. The paper deals with both the overall outcome and the sequence of game by game outcomes. A method for predicting the overall outcome is advanced and illustrated. Methods for predicting game by game outcomes are examined and compared according to strengths and weaknesses. The analysis is supported by the data on World Championship matches since their beginning 125 years ago.

1. INTRODUCTION

In this paper we present a theory based on the analytic hierarchy process to predict the outcome of World Chess Championship matches. The main idea underlying the theory is the modeling of the behavior and technical ability of the contenders in terms of the factors which are deemed relevant. The importance of behavior in chess competition has been highlighted in the literature. A chess master [2] writes, "Top flight chess is as much psychological battle as technical ability." Another characterization of the game [6] is given in more vivid language:

> The essential quality of a high-level player is a kind of enjoyment of a very intense, physically and mentally exhausting struggle. Chess players are not necessarily mathematical, or artistic, or more intelligent,.... You attack a person's psychological weaknesses. You put him under tremendous strain, push him to where he consumes his energy where he gets exhausted,.... When he reaches a point of demoralization, a player can crash, go to pieces, lose.

In general, mathematical analyses of indoor games have been made purely in terms of the strategies of the players without consideration of their

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behavior. To be accurate, prediction must deal with the inputs of a system, its actual operation, and its outputs. The problem is: given certain inputs, be they technical or behavioral, what is the output?

In chess, the raw input into the game is the experience and know-how of the players in manipulating or transforming the system (i.e., the chessboard and its psychological environment) with respect to an output, which in practice is a win, a draw, or a loss.

The first task is to assess the quality of the input by deriving a relative index of power of the players considering all relevant characteristics. The second task is to use this power index to assess the kind of output it would produce over a set of several encounters in a match.

An objective in predicting the outcome of a chess match is to identify the winner. A more ambitious objective is to predict the total number of games drawn or won by each player. A third objective is to test the stability of the outcome by means of sensitivity analysis with variations in judgments about the players' abilities.

We note that the statistics of a player's past wins, losses, and draws against a variety of opponents are not very useful for predicting how he would fare against a champion challenger. Assessment of the outcome of a match must be made in terms of the competition between particular players. The output must be evaluated both in terms of the power of each player, and in terms of how he perceives the abilities of his opponent [14].

2. OUTLINE OF THE METHOD

To determine the most relevant factors involved in chess we first examined the literature [3–5] and then sent out a questionnaire to grand masters. The questionnaire included both the technical (T) and behavioral (B) characteristics of chess players listed in Table 1. Each factor should be interpreted as falling only in the category indicated, although the approach is independent of how they are classified and some of them might be listed under both categories. These factors are then used to construct the hierarchy of Figure 1.

Next we compute the relative power of the players in the match. To do this we carried out pairwise comparisons of the factors in the hierarchy according to the analytic hierarchy process. The scale for making comparisons in that process is shown in Figure 2. The questionnaire, shown in Figure 3, asked grand masters to make qualitative pairwise comparisons of the players with respect to each criterion. The two results were then combined to obtain a power index for the two contenders. Of course, the grand masters did not know the outcome for any two closely matched players. They had to guess who qualified best according to each factor.

TABLE 1

DEFINITIONS OF CHESS FACTORS

T	(1)	*Calculation* (C): The ability of a player to evaluate different alternatives or strategies in light of prevailing situations.
B	(2)	*Ego* (E): The image a player has of himself as to his general abilities and qualification and his desire to win.
T	(3)	*Experience* (EX): A composite of the versatility of opponents faced before, the strength of the tournaments participated in, and the time of exposure to a rich variety of chess players.
B	(4)	*Gamesmanship* (G): The capability of a player to influence his opponent's game by destroying his concentration and self-confidence.
T	(5)	*Good health* (GH): Physical and mental strength to withstand pressure and provide endurance.
B	(6)	*Good nerves and the will to win* (GNWW): The attitude of steadfastness that ensures a player's health perspective while the going gets tough. He keeps in mind that the situation involves two people and that if he holds out the tide may go in his favor.
T	(7)	*Imagination* (IM): Ability to perceive and improvise good tactics and strategies.
T	(8)	*Intuition* (IN): Ability to guess the opponent's intentions.
T	(9)	*Game aggressiveness* (GA): The ability to exploit the opponent's weaknesses and mistakes to one's advantage. Occasionally referred to as "killer instinct."
T	(10)	*Long range planning* (LRP): The ability of a player to foresee the outcome of a certain move, set up desired situations that are more favorable, and work to alter the outcome.
T	(11)	*Memory* (M): Ability to remember previous games.
B	(12)	*Personality* (P): Manners and emotional strength, and their effects on the opponent in playing the game and on the player in keeping his wits.
T	(13)	*Preparation* (PR): Study and review of previous games and ideas.
T	(14)	*Quickness* (Q): The ability of a player to see clearly the heart of a complex problem.

TABLE 1—*Continued*

T	(15)	*Relative youth* (RY): The vigor, aggressiveness, and daring to try new ideas and situations, a quality usually attributed to young age.
T	(16)	*Seconds* (S): The availability of other experts to help one to analyze strategies between games.
B	(17)	*Stamina* (ST): Physical and psychological ability of a player to endure fatigue and pressure.
T	(18)	*Technique* (T): Ability to use and respond to different openings, improvise middle game tactics, and steer the game to a familiar ground to one's advantage.

In chess, the possibility of a draw complicates the calculation of the outcome. The power index which is evaluated in terms of technical and behavioral factors only gives the relative strength of the players. It divides each game into two parts: How much goes to one player and how much goes to the other. It says nothing about draws. Our prediction must also cover the number of games drawn. This is done by assessing the disposition or perception of the players towards each other. An experienced contestant will go for a win or a draw in a game according to his perception of the strength of his opponent. If he expects his opponent to be strong, he is more likely to allow the possibility of a draw or a loss than a win. If he plays a weak player, with less reservation, he will go for a win.

Thus, prediction of the outcome of chess matches is determined by two parameters:

(1) the relative strength of the players derived from their technical and behavioral characteristics as assessed by expert judgment such as those of grand masters, and
(2) their attitude towards winning, drawing, or losing against the opponent.

To predict the actual chain of wins, draws, and losses is a more difficult task, subject to lack of knowledge of the opening moves to be used and the impact of earlier wins and losses under similar conditions on the psychology of the players in the present game. Still, from the record of championship matches since 1858 and the corresponding relative power of the players, we have developed a method of predicting games that appears to be better than making random guesses.

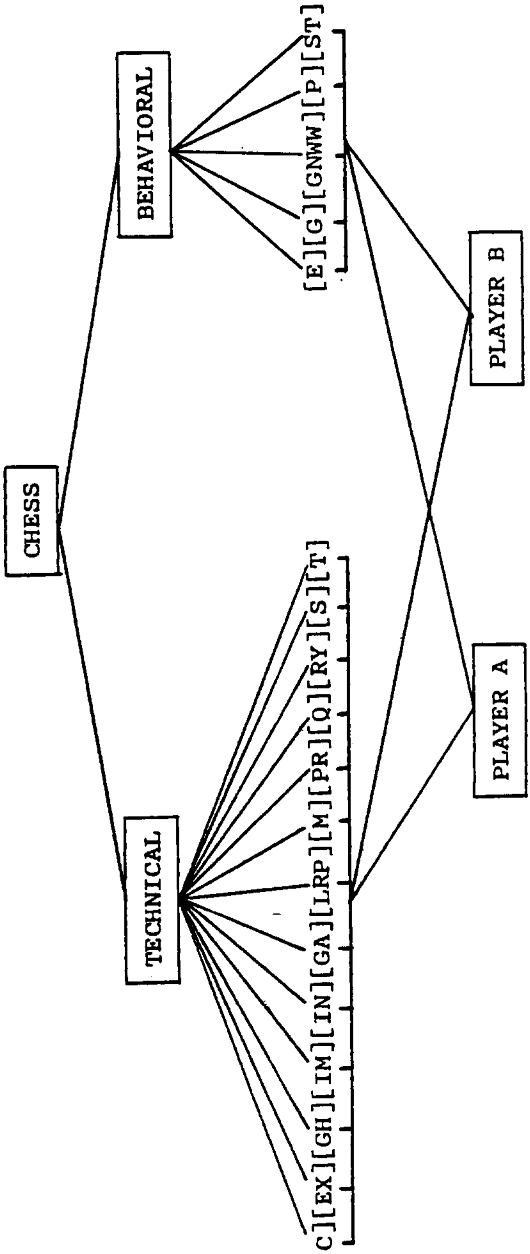

FIG. 1. Factors that influence a chess match.

Intensity of Importance	Definition	Explanation
1	Equal importance	Two activities contribute equally to the objective
3	Moderate importance of one over another	Experience and judgment slightly favor one activity over another
5	Essential or strong importance	Experience and judgment strongly favor one activity over another
7	Demonstrated importance	An activity is strongly favored and its dominance is demonstrated in practice
9	Extreme importance	The evidence favoring one activity over another is of the highest possible order of affirmation
2, 4, 6, 8	Intermediate values between the two adjacent judgments	Compromise is needed
Reciprocals of above nonzero	If activity i has one of the above nonzero numbers assigned to it when compared with activity j, then j has the reciprocal value when compared with i	

FIG. 2. The scale and its description.

2. THE RELATIVE POWER OF THE PLAYERS AND THEIR EXPECTATIONS

Denote the two players in a match by A and B, and assume that player A is the winner. In chess, points are awarded as follows: one point for a win, half a point for a draw, and zero for a loss. Let P_{AW}, P_D, and P_{AL} be the proportion of points accumulated by player A by winning, drawing and losing games, respectively. Let P_{BW}, P_D, and P_{BL} be the corresponding proportions for B. Let S_A and S_B be the relative strengths of player A and player B, respectively, where $S_A + S_B = 1$. Also let N_A, N_D, and N_B denote the numbers of games won, drawn, and lost by player A. If n is the total number of games played then $N_A + N_D + N_B = n$. To predict the outcome of a match we first estimate n, from which we find N_A and N_B. There are situations in which we estimate n and N_B because the value of N_A is fixed by the rules at 6.